

Existence of guided waves due to a lineic perturbation of a 3D periodic medium

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Abstract

In this note, we exhibit a three dimensional structure that permits to guide waves. This structure is obtained by a geometrical perturbation of a 3D periodic domain that consists of a three dimensional grating of equi-spaced thin pipes oriented along three orthogonal directions. Homogeneous Neumann boundary conditions are imposed on the boundary of the domain. The diameter of the section of the pipes, of order $\varepsilon > 0$, is supposed to be small. We prove that, for ε small enough, shrinking the section of one line of the grating by a factor of $\sqrt{\mu}$ ($0 < \mu < 1$) creates guided modes that propagate along the perturbed line. Our result relies on the asymptotic analysis (with respect to ε) of the spectrum of the Laplace-Neumann operator in this structure. Indeed, as ε tends to 0, the domain tends to a periodic graph, and the spectrum of the associated limit operator can be computed explicitly.

Keywords : guided waves, periodic media, spectral theory.

AMS codes : 78M35, 35J05, 58C40

1 Statement of the problem

Let ω_1 , ω_2 and ω_3 be three Lipschitz bounded domains of \mathbb{R}^2 of same area ($|\omega_1| = |\omega_2| = |\omega_3|$) containing the origin $(0,0)$, let $\varepsilon > 0$ be a parameter (that is going to be small), and let a_1 , a_2 and a_3 be three positive real numbers. We denote by $(\mathbf{e}_i)_{i \in \{1,2,3\}}$, the standard basis of \mathbb{R}^3 . For any $(k, \ell) \in \mathbb{Z}^2$, we consider the three dimensional domain $D_{k,\ell,3}^\varepsilon$ defined by

$$D_{k,\ell,3}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_1 - a_1 k)/\varepsilon, (x_2 - a_2 \ell)/\varepsilon) \in \omega_3\},$$

which is an unbounded cylinder of constant cross section $\varepsilon \omega_3$. It is infinite along the \mathbf{e}_3 direction (invariant with respect to x_3) and contains the point $(a_1 k, a_2 \ell, 0)$. Similarly, for any $(k, \ell) \in \mathbb{Z}^2$, we define the domains

$$D_{k,\ell,1}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_2 - a_2 k)/\varepsilon, (x_3 - a_3 \ell)/\varepsilon) \in \omega_1\},$$

$$D_{k,\ell,2}^\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } ((x_1 - a_1 k)/\varepsilon, (x_3 - a_3 \ell)/\varepsilon) \in \omega_2\},$$

and we consider the periodic domain Ω_ε given by

$$\Omega_\varepsilon = \bigcup_{i \in \{1,2,3\}} \bigcup_{(k,\ell) \in \mathbb{Z}^2} D_{k,\ell,i}^\varepsilon. \quad (1)$$

The domain Ω_ε is a three dimensional grating of equi-spaced parallel pipes (of constant cross section) oriented along the three orthogonal directions \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 . It is a_j -periodic with respect to x_j , $j = 1, 2, 3$. Moreover, the points $(ka_1, \ell a_2, ma_3)$, $(k, \ell, m) \in \mathbb{Z}^3$, belong to Ω_ε .

In order to create guided modes, we introduce a linear defect (see [1]-[5]-[2]) in the periodic structure by modifying the section size of one pipe of the grating (it is conjectured that guided modes cannot appear in the purely periodic structure, see [4] for the proof in the case of a symmetric medium). More precisely, we assume that the domain $D_{0,0,3}^\varepsilon$ is replaced with the domain

$$D_{0,0,3}^{\varepsilon,\mu} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that } (x_1/(\sqrt{\mu}\varepsilon), x_2/(\sqrt{\mu}\varepsilon)) \in \omega_3\},$$

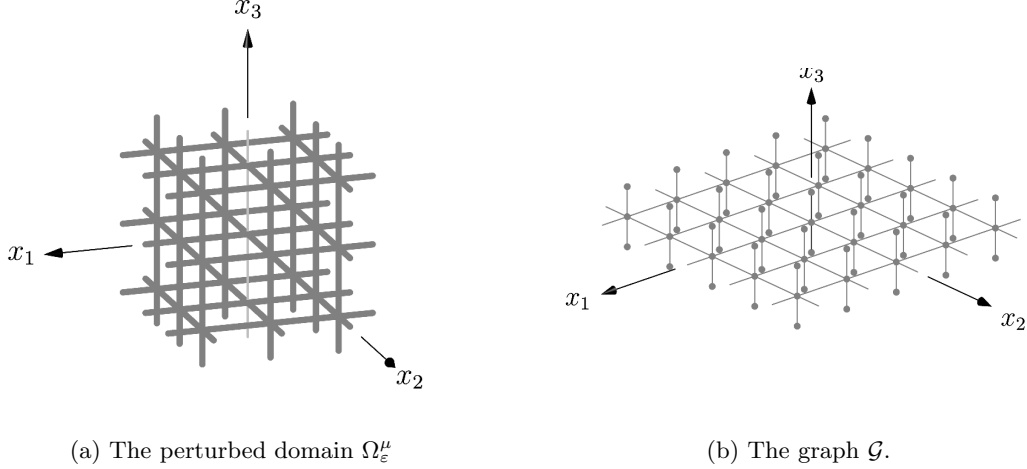


Figure 1: Illustration of the perturbed periodic domain Ω_ε^μ and the limit graph \mathcal{G} .

where μ is a positive parameter. In other words, we enlarge ($\mu > 1$) or shrink ($0 < \mu < 1$) the section of one pipe of the domain by a factor μ (see Fig 1a). The corresponding perturbed domain is denoted by Ω_ε^μ . Its precise definition is given by

$$\Omega_\varepsilon^\mu = \left(\bigcup_{i \in \{1,2\}} \bigcup_{(k,\ell) \in \mathbb{Z}^2} D_{k,\ell,i}^\varepsilon \right) \cup \left(\bigcup_{(k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} D_{k,\ell,3}^\varepsilon \right) \cup D_{0,0,3}^{\varepsilon,\mu}. \quad (2)$$

Ω_ε^μ is still a_3 -periodic with respect to x_3 . However, the presence of the perturbed pipe $D_{0,0,3}^{\varepsilon,\mu}$ breaks the periodicity with respect to x_1 and x_2 . We emphasize that the domain Ω_ε^μ (as well as Ω_ε) tends to a 3D periodic graph as ε tends to 0.

We look for guided modes, i.e. solutions to the wave equation $\partial_t^2 u - \Delta u = 0$ in Ω_ε^μ , satisfying homogeneous Neumann boundary conditions on $\partial\Omega_\varepsilon^\mu$ (see [7] for the investigation of the Dirichlet case), that propagate along the defect pipe $D_{0,0,3}^{\varepsilon,\mu}$ (i.e. in the \mathbf{e}_3 direction) but stay confined in the transversal directions. More precisely, denoting by B_ε^μ the restriction of the domain Ω_ε^μ to the band $|x_3| < a_3/2$,

$$B_\varepsilon^\mu = \{(x_1, x_2, x_3) \in \Omega_\varepsilon^\mu \text{ such that } |x_3| < a_3/2\}, \quad (3)$$

we search solutions of the form $u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3)e^{i\omega t - \beta x_3}$, where β is a real parameter and $v(x_1, x_2, x_3) \in L_2(B_\varepsilon^\mu)$ is an a_3 -periodic function in x_3 . In fact, it is easily seen that the β -quasiperiodic function $v(x_1, x_2, x_3)e^{-i\beta x_3}$ is an eigenfunction of the operator

$$A_\varepsilon^\mu(\beta) : D(A_\varepsilon^\mu(\beta)) \subset L_2(B_\varepsilon^\mu) \rightarrow L_2(B_\varepsilon^\mu), \quad A_\varepsilon^\mu(\beta) = -\Delta u \text{ in } B_\varepsilon^\mu, \quad (4)$$

with $D(A_\varepsilon^\mu(\beta)) = \{u \in H_\Delta^1(B_\varepsilon^\mu), u|_{\Sigma^+} = e^{-i\beta} u|_{\Sigma^-}, \partial_{x_3} u|_{\Sigma^+} = e^{-i\beta} \partial_{x_3} u|_{\Sigma^-}, \partial_n u|_{\partial B_\varepsilon^\mu \setminus \Sigma^\pm} = 0\}$, where

$$H_\Delta^1(B_\varepsilon^\mu) = \{u \in H^1(B_\varepsilon^\mu), \text{ s.t. } \Delta u \in L_2(B_\varepsilon^\mu)\} \text{ and } \Sigma^\pm = \{(x_1, x_2, x_3) \in \partial B_\varepsilon^\mu, x_3 = \pm a_3/2\}.$$

To study the spectral properties of $A_\varepsilon^\mu(\beta)$, we investigate its (formal) limit $\mathcal{A}^\mu(\beta)$ as ε tends to 0. The operator $\mathcal{A}^\mu(\beta)$ is defined on the limit graph \mathcal{G} (see Fig. 1b) and its spectrum can be explicitly computed. In particular, its spectrum has infinitely many gaps (Lemma 2.1), i.e. open intervals $(a, b) \subset \mathbb{R}$ such that the intersection of $[a, b]$ with the spectrum is reduced to $\{a, b\}$. Moreover, for $\mu < 1$, there is at least one eigenvalue in each gap (Lemma 2.5). Since, in addition, for $\varepsilon > 0$ sufficiently small, the spectrum of $A_\varepsilon^\mu(\beta)$ is close to the spectrum of $\mathcal{A}^\mu(\beta)$, the existence of guided modes is guaranteed (Theorem 3.1).

2 The spectrum of the limit operator $\mathcal{A}^\mu(\beta)$

2.1 Definition of the limit operator $\mathcal{A}^\mu(\beta)$

The limit operator $\mathcal{A}^\mu(\beta)$ is defined on the infinite periodic graph $\mathcal{G} = \bigcap_{\varepsilon>0} B_\varepsilon^\mu$ obtained as the limit of B_ε^μ as ε tends to 0: \mathcal{G} is made of the vertices $\{v_{k,\ell} = (ka_1, \ell a_2, 0), v_{k,\ell}^\pm = (ka_1, \ell a_2, \pm a_3/2), (k, \ell) \in \mathbb{Z}^2\}$ connected by the edges

$$\{e_{k+1/2,\ell} = (v_{k,\ell}, v_{k+1,\ell}), e_{k,\ell+1/2} = (v_{k,\ell}, v_{k,\ell+1}), e_{k,\ell}^\pm = (v_{k,\ell}, v_{k,\ell}^\pm), (k, \ell) \in \mathbb{Z}^2\}.$$

It is a_1 -periodic with respect to x_1 and a_2 -periodic with respect to x_2 (see Fig. 1b).

For any function u defined on \mathcal{G} , we denote by $\mathbf{u}_{k,\ell}$ (resp. $\mathbf{u}_{k,\ell}^\pm$) its value at the vertex $v_{k,\ell}$ (resp. $v_{k,\ell}^\pm$). The restriction of u to the edge $e_{k+1/2,\ell}$ (resp. $e_{k,\ell+1/2}$ and $e_{k,\ell}^\pm$) is denoted by $u_{k+1/2,\ell}(x_1)$ (resp. $u_{k,\ell+1/2}(x_2)$ and $u_{k,\ell}^\pm(x_3)$).

The definition of $\mathcal{A}^\mu(\beta)$ also requires the introduction of the function spaces $L_2^\mu(\mathcal{G})$ and $H^2(\mathcal{G})$ defined as

$$L_2^\mu(\mathcal{G}) = \left\{ u : \|u\|_{L_2^\mu(\mathcal{G})} < +\infty \right\}, \quad H^2(\mathcal{G}) = \left\{ u \in C(\mathcal{G}) : \|u\|_{H^2(\mathcal{G})} < +\infty \right\}, \quad (5)$$

where,

$$\|u\|_{L_2^\mu(\mathcal{G})}^2 = \sum_{(k,\ell) \in \mathbb{Z}^2} \left(w_{k,\ell}^\mu \sum_{\pm} \|u_{k,\ell}^\pm\|_{L_2(e_{k,\ell}^\pm)}^2 + \|u_{k+\frac{1}{2},\ell}\|_{L_2(e_{k+\frac{1}{2},\ell})}^2 + \|u_{k,\ell+\frac{1}{2}}\|_{L_2(e_{k,\ell+\frac{1}{2}})}^2 \right), \quad (6)$$

$$\|u\|_{H^2(\mathcal{G})}^2 = \sum_{(k,\ell) \in \mathbb{Z}^2} \left(\sum_{\pm} \|u_{k,\ell}^\pm\|_{H^2(e_{k,\ell}^\pm)}^2 + \|u_{k+\frac{1}{2},\ell}\|_{H^2(e_{k+\frac{1}{2},\ell})}^2 + \|u_{k,\ell+\frac{1}{2}}\|_{H^2(e_{k,\ell+\frac{1}{2}})}^2 \right), \quad (7)$$

and, for any $(k, \ell) \in \mathbb{Z}^2$, $w_{k,\ell}^\mu$ is the weight coefficient equal to μ for $k = \ell = 0$ and 1 otherwise.

The unbounded limit operator in $L_2^\mu(\mathcal{G})$ has domain

$$D(\mathcal{A}^\mu(\beta)) = \left\{ u \in H^2(\mathcal{G}) : \quad \forall (k, \ell) \in \mathbb{Z}^2, \quad \mathbf{u}_{k,\ell}^+ = e^{-i\beta} \mathbf{u}_{k,\ell}^-, \quad (u_{k,\ell}^+)'(a_3/2) = e^{-i\beta} (u_{k,\ell}^-)'(-a_3/2), \right. \\ \left. u'_{k+\frac{1}{2},\ell}(ka_1) - u'_{k-\frac{1}{2},\ell}(ka_1) + u'_{k,\ell+\frac{1}{2}}(\ell a_2) - u'_{k,\ell-\frac{1}{2}}(\ell a_2) + w_{k,\ell}^\mu \left((u_{k,\ell}^+)'(0) - (u_{k,\ell}^-)'(0) \right) = 0 \right\}, \quad (8)$$

and is defined by

$$\forall u \in D(\mathcal{A}^\mu(\beta)), \quad \mathcal{A}^\mu(\beta)u = -u'' \quad \text{on any edge of the graph } \mathcal{G}. \quad (9)$$

The functions of $D(\mathcal{A}^\mu(\beta))$ are continuous on \mathcal{G} and β quasi-periodic. Moreover, they satisfy the Kirchhoff conditions (8) that enforce the weighted sum of the outward derivatives of u to vanish at each vertex $v_{k,\ell}$ ($(k, \ell) \in \mathbb{Z}^2$). We point out that the perturbation, which results from a geometrical modification of the domain for the problem (4), is taken into account at the limit by means of the Kirchhoff condition (8) at the vertex $v_{0,0}$ ($w_{0,0}^\mu = \mu$). The formal derivation of the limit model can be found in [6]. It is easily verified that the operator $\mathcal{A}^\mu(\beta)$ is self-adjoint (for the weighted scalar product associated with (6)), see also [3]. The objective of the following two sections is to study the spectrum of $\mathcal{A}^\mu(\beta)$.

2.2 Characterization and properties of the essential spectrum of $\mathcal{A}^\mu(\beta)$

By a compact perturbation argument, one can prove that $\sigma_{ess}(\mathcal{A}^\mu(\beta)) = \sigma(\mathcal{A}(\beta))$, where $\mathcal{A}(\beta) = \mathcal{A}^1(\beta)$ is the purely periodic operator corresponding to $\mathcal{A}^\mu(\beta)$ for $\mu = 1$. The computation of its spectrum relies on the Floquet-Bloch theory (see [9]). More precisely, we can prove that $\lambda = \omega^2 \in \sigma(\mathcal{A}(\beta))$ if and only if either $\omega = 0$ and $\beta = 0$ or $\omega \neq 0$ and there exists $(k_1, k_2) \in [0, \pi]^2$ such that

$$\sin(\omega a_2) \sin(\omega a_3) (\cos(\omega a_1) - \cos k_1) + \sin(\omega a_3) \sin(\omega a_1) (\cos(\omega a_2) - \cos k_2) \\ + \sin(\omega a_1) \sin(\omega a_2) (\cos(\omega a_3) - \cos \beta) = 0. \quad (10)$$

Based on the previous characterization, we prove that the operator $\mathcal{A}(\beta)$ has a countable infinity of gaps that can be separated into three categories (see [10] for the proof):

Lemma 2.1 *The following properties hold :*

1- $\sigma_1 \cup \sigma_2 \cup \sigma_3 \subset \sigma(\mathcal{A}(\beta))$, where

$$\sigma_i = \left\{ (\pi n/a_i)^2, n \in \mathbb{Z} \right\} \text{ for } i \in \{1, 2\}, \text{ and } \sigma_3 = \left\{ ((\pm\beta + 2\pi n)/a_3)^2, n \in \mathbb{Z} \right\}.$$

2- For any $\beta \in [0, \pi]$, the operator $\mathcal{A}(\beta)$ has infinitely many gaps whose ends tend to infinity.

3- Let $\mathcal{W}(\beta) = \{\pi n/a_3, n \in \mathbb{N}^*\}$ if $\beta \notin \{0, \pi\}$ and $\mathcal{W}(\beta) = \{\beta/a_3 + (2n+1)\pi/a_3, n \in \mathbb{N}^*\}$ if $\beta \in \{0, \pi\}$.

If an interval (ω_b^2, ω_t^2) is a spectral gap of $\mathcal{A}(\beta)$, then, one of the following possibilities holds:

- (i) $\omega_b^2 \notin \sigma_1 \cup \sigma_2$, $\omega_t^2 \notin \sigma_1 \cup \sigma_2$, and there is a unique $\omega_0 \in (\omega_b, \omega_t) \cap \mathcal{W}(\beta)$.
- (ii) $\omega_b^2 \in \sigma_1 \cup \sigma_2$, $\omega_t^2 \notin \sigma_1 \cup \sigma_2$ and $\mathcal{W}(\beta) \cap (\omega_b, \omega_t) = \emptyset$.
- (iii) $\omega_b^2 \notin \sigma_1 \cup \sigma_2$, $\omega_t^2 \in \sigma_1 \cup \sigma_2$ and $\mathcal{W}(\beta) \cap (\omega_b, \omega_t) = \emptyset$.

2.3 Computation of the discrete spectrum

Let us now determine the discrete spectrum of $\mathcal{A}^\mu(\beta)$. If $\lambda = \omega^2$ is an eigenvalue of $\mathcal{A}^\mu(\beta)$, then the corresponding eigenfunction $u \in D(\mathcal{A}^\mu(\beta))$ satisfies the linear differential equation $u'' + \omega^2 u = 0$ on each edge of the graph \mathcal{G} . Solving explicitly this equation (on each edge), taking into account the quasi-periodicity of u and the Kirchhoff conditions (8), we can replace the eigenvalue problem $\mathcal{A}^\mu(\beta)u = \lambda u$ with a set of finite differences equations for $(\mathbf{u}_{\ell,k})_{(\ell,k) \in \mathbb{Z}^2}$:

Lemma 2.2 *Assume that $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$. $u \in D(\mathcal{A}^\mu(\beta))$ is an eigenfunction of $\mathcal{A}^\mu(\beta)$ if and only if $(\mathbf{u}_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}$ belongs to $\ell_2(\mathbb{Z}^2)$ and satisfies*

$$\begin{cases} \frac{\mathbf{u}_{k+1,\ell}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{k-1,\ell}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{k,\ell+1}}{\sin(\omega a_2)} + \frac{\mathbf{u}_{k,\ell-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)\mathbf{u}_{k,\ell} = 0, & \forall (k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\ \frac{\mathbf{u}_{1,0}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{-1,0}}{\sin(\omega a_1)} + \frac{\mathbf{u}_{0,1}}{\sin(\omega a_2)} + \frac{\mathbf{u}_{0,-1}}{\sin(\omega a_2)} - 2g_\beta(\omega)\mathbf{u}_{0,0} = 2(\mu-1)\frac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}\mathbf{u}_{0,0}, \end{cases} \quad (11)$$

where we have defined $g_\beta(\omega) = \frac{1}{\tan(\omega a_1)} + \frac{1}{\tan(\omega a_2)} + \frac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}$.

As well-known, finite difference schemes may be investigated using the discrete Fourier transform

$$\mathcal{F} : \mathbf{v} = (\mathbf{v}_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2} \mapsto \mathcal{F}(\mathbf{v}) = \widehat{\mathbf{v}}, \quad \widehat{\mathbf{v}}(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}} e^{i(k\xi + \ell\eta)} \mathbf{v}_{k,\ell}, \quad (\xi, \eta) \in [0, 2\pi]^2, \quad (12)$$

where \mathcal{F} is an isometry between $\ell_2(\mathbb{Z}^2)$ and $L_2((0, 2\pi)^2)$. This, together with Lemma 2.2, provides the following characterization for the discrete spectrum of $\mathcal{A}^\mu(\beta)$:

Lemma 2.3 *Assume that $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$. $u \in D(\mathcal{A}^\mu(\beta))$ is an eigenfunction of $\mathcal{A}^\mu(\beta)$ if and only if the discrete Fourier transform $\widehat{\mathbf{u}}$ of $(\mathbf{u}_{\ell,k})_{(\ell,k) \in \mathbb{Z}^2}$ belongs to $L_2((0, 2\pi)^2)$ and satisfies*

$$(f(\xi, \eta, \omega) - \phi_\beta(\omega)) \widehat{\mathbf{u}}(\xi, \eta) = (\mu-1)\phi_\beta(\omega)\mathbf{u}_{0,0}. \quad (13)$$

where $\phi_\beta(\omega) = \frac{\cos(\omega a_3) - \cos\beta}{\sin(\omega a_3)}$ and $f(\xi, \eta, \omega) = \frac{\cos\xi - \cos(\omega a_1)}{\sin(\omega a_1)} + \frac{\cos\eta - \cos(\omega a_2)}{\sin(\omega a_2)}$.

Under the assumption of Lemma 2.3, (10) can be written as $f(\xi, \eta, \omega) - \phi_\beta(\omega) = 0$. It follows that $\lambda = \omega^2$ does not belong to $\sigma_{ess}(\mathcal{A}^\mu(\beta))$ if and only if, for any $(\xi, \eta) \in [0, 2\pi]^2$, $\phi_\beta(\omega) - f(\xi, \eta, \omega)$ does not vanish. As a consequence, as soon as $\lambda = \omega^2 \notin \sigma_{ess}(\mathcal{A}^\mu(\beta))$, the function $(\xi, \eta) \mapsto \phi_\beta(\omega)/(\phi_\beta(\omega) - f(\xi, \eta, \omega))$ is continuous and bounded. Then, the inverse discrete Fourier transform can be applied to (13) to obtain

$$\mathbf{u}_{k,\ell} = \frac{(1-\mu)\mathbf{u}_{0,0}}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi, \eta, \omega)} e^{-i(k\xi + \ell\eta)} d\xi d\eta, \quad \forall (k, \ell) \in \mathbb{Z}^2.$$

Writing the previous relation for $k = \ell = 0$ yields the following criterion of existence of an eigenvalue:

Lemma 2.4 Assume that $\omega \notin \{\pi\mathbb{Z}/a_1\} \cup \{\pi\mathbb{Z}/a_2\} \cup \{\pi\mathbb{Z}/a_3\}$ and that $\lambda = \omega^2 \notin \sigma_{\text{ess}}(\mathcal{A}^\mu(\beta))$. Then, λ is an eigenvalue of $\mathcal{A}^\mu(\beta)$ if and only if

$$\mu = 1 - F_\beta(\omega) \quad \text{where} \quad F_\beta(\omega) = \left(\frac{1}{4\pi^2} \int_{(0,2\pi)^2} \frac{\phi_\beta(\omega)}{\phi_\beta(\omega) - f(\xi, \eta, \omega)} d\xi d\eta \right)^{-1}. \quad (14)$$

The study of the behavior of the function F_β leads to the existence of at least one eigenvalue in each gap of $\mathcal{A}^\mu(\beta)$ as soon as $\mu < 1$, the minimal number of eigenvalues in each gap depending on the type of gaps (cf. Lemma. 2.1-3 for the classification):

Lemma 2.5 For $\mu > 1$, the operator $\mathcal{A}^\mu(\beta)$ has no eigenvalue. For $0 < \mu < 1$, let (ω_b^2, ω_t^2) be a spectral gap of the operator $\mathcal{A}^\mu(\beta)$:

- (a) If (ω_b^2, ω_t^2) is a gap of type (i), then $\mathcal{A}^\mu(\beta)$ has at least two eigenvalues $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$ that satisfy $\omega_b < \omega_1 < \omega_0 < \omega_2 < \omega_t$ (see Lemma. 2.1-3 for the definition of ω_0).
- (b) If (ω_b^2, ω_t^2) is a gap of type (ii) or (iii), then $\mathcal{A}^\mu(\beta)$ has at least one eigenvalue $\lambda_1 = \omega_1^2$ such that $\omega_b < \omega_1 < \omega_t$.

The sketch of the proof of the previous lemma is the following, a complete proof being available in [10] (Theorem 5.2.1): First, one can verify that $F_\beta(\omega) \geq 0$ in any gap, which, together with (14) proves that $\mathcal{A}^\mu(\beta)$ has no eigenvalue for $\mu > 1$. Then, if (ω_b^2, ω_t^2) is a gap of type (i), one can show that

$$\lim_{\omega \rightarrow \omega_b^+} (1 - F_\beta(\omega)) = \lim_{\omega \rightarrow \omega_t^-} (1 - F_\beta(\omega)) = 1 \text{ and that } \lim_{\omega \rightarrow \omega_0} (1 - F_\beta(\omega)) = 0.$$

By continuity of F_β inside the gap, (a) directly results from the intermediate value theorem and (14). If (ω_b^2, ω_t^2) is a gap of type (ii), the intermediate value theorem also permits us to conclude since

$$\lim_{\omega \rightarrow \omega_b} (1 - F_\beta(\omega)) \leq 0 \text{ and } \lim_{\omega \rightarrow \omega_t^-} (1 - F_\beta(\omega)) = 1.$$

A similar argument works for a gap of type (iii).

3 Guided modes for the operator $A_\varepsilon^\mu(\beta)$: an asymptotic result

Finally, thanks to the general result [8] (Theorem 2.13 convergence of the spectrum of $A_\varepsilon^\mu(\beta)$ toward the spectrum of $\mathcal{A}^\mu(\beta)$), we can prove the following result of existence of eigenvalue for the operator $A_\varepsilon^\mu(\beta)$:

Theorem 3.1 Let $\mu \in (0, 1)$, (λ_b, λ_t) be a spectral gap of the operator $\mathcal{A}^\mu(\beta)$ and $\lambda_0 \in (\lambda_b, \lambda_t)$ be a (simple) eigenvalue of this operator. Then, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_\varepsilon^\mu(\beta)$ has an eigenvalue λ_ε inside a spectral gap $(\lambda_b^\varepsilon, \lambda_t^\varepsilon)$. Moreover, $\lambda_\varepsilon = \lambda_0 + O(\sqrt{\varepsilon})$.

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